

WEIERSTRASS REPRESENTATION FOR TIMELIKE MINIMAL SURFACES IN MINKOWSKI 3-SPACE

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ABSTRACT. Using techniques of integrable systems, we study a Weierstraß representation formula for timelike surfaces with prescribed mean curvature in Minkowski 3-space. It is shown that timelike minimal surfaces are obtained by integrating a pair of Lorentz holomorphic and Lorentz antiholomorphic null curves in Minkowski 3-space. The relationship between timelike minimal surfaces and bosonic Nambu-Goto string worldsheets in spacetime is also discussed in the appendix.

1. PRELIMINARIES

In this section, we review some basics on the geometry of timelike surfaces in Minkowski 3-space.

Let \mathbb{E}_2^4 be the semi-Euclidean 4-space with rectangular coordinates x_0, x_1, x_2, x_3 and the semi-Riemannian metric $\langle \cdot, \cdot \rangle$ of signature $(-, -, +, +)$ given by the quadratic form

$$ds^2 = -dx_0^2 - dx_1^2 + dx_2^2 + dx_3^2.$$

The semi-Euclidean 4-space \mathbb{E}_2^4 is identified with the linear space $M_2\mathbb{R}$ of all 2×2 real matrices via the correspondence

$$(1) \quad \mathbf{u} = (x_0, x_1, x_2, x_3) \longleftrightarrow \begin{pmatrix} x_0 + x_3 & x_1 + x_2 \\ -x_1 + x_2 & x_0 - x_3 \end{pmatrix}.$$

This identification is an isometry, since

$$\langle \mathbf{u}, \mathbf{v} \rangle = \frac{1}{2} \{ \text{tr}(\mathbf{u}\mathbf{v}) - \text{tr}(\mathbf{u})\text{tr}(\mathbf{v}) \}, \quad \mathbf{u}, \mathbf{v} \in M_2\mathbb{R}.$$

In particular, $\langle \mathbf{u}, \mathbf{u} \rangle = -\det \mathbf{u}$. The standard basis e_0, e_1, e_2, e_3 for \mathbb{E}_2^4 is identified with the matrices

$$\mathbf{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{i} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \mathbf{j}' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \mathbf{k}' = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

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i.e.

$$(2) \quad (x_0, x_1, x_2, x_3) \longleftrightarrow x_0 \mathbf{1} + x_1 \mathbf{i} + x_2 \mathbf{j}' + x_3 \mathbf{k}'.$$

Note that the 2×2 matrices $x_0 \mathbf{1} + x_1 \mathbf{i} + x_2 \mathbf{j}' + x_3 \mathbf{k}'$ form the algebra \mathbb{H}' of *para-quaternions* or *split-quaternions*. The basis $\{\mathbf{1}, \mathbf{i}, \mathbf{j}', \mathbf{k}'\}$ satisfies the following relation:

$$\begin{aligned} \mathbf{i}^2 &= -\mathbf{1}, \quad \mathbf{j}'^2 = \mathbf{k}'^2 = \mathbf{1}, \\ \mathbf{i}\mathbf{j}' &= -\mathbf{j}'\mathbf{i} = \mathbf{k}', \quad \mathbf{j}'\mathbf{k}' = -\mathbf{k}'\mathbf{j}' = -\mathbf{i}, \quad \mathbf{k}'\mathbf{i} = -\mathbf{i}\mathbf{k}' = \mathbf{j}'. \end{aligned}$$

The imaginary split-quaternions

$$\text{Im}\mathbb{H}' = \{x_1 \mathbf{i} + x_2 \mathbf{j}' + x_3 \mathbf{k}' : x_i \in \mathbb{R}, i = 1, 2, 3\}$$

is identified with Minkowski 3-space \mathbb{E}_1^3 .

Let M be a connected orientable 2-manifold and $\varphi : M \longrightarrow \mathbb{E}_1^3$ an immersion. The immersion φ is said to be *timelike* if the induced metric I on M is Lorentzian. The induced Lorentzian metric I determines a Lorentz conformal structure \mathcal{C}_I on M . Let (x, y) be a Lorentz isothermal coordinate system with respect to the conformal structure \mathcal{C}_I . Then the first fundamental form I is written in terms of (x, y) as

$$I = e^\omega (-dx^2 + dy^2).$$

(M, \mathcal{C}_I) is said to be a *Lorentz surface*.

Let N be a unit normal vector field of M . Then

$$\langle N, N \rangle = 1, \quad \langle \varphi_x, N \rangle = \langle \varphi_y, N \rangle = 0.$$

The *conformality conditions* are given by

$$(3) \quad \begin{cases} \langle \varphi_x, \varphi_y \rangle = 0, \\ -\langle \varphi_x, \varphi_x \rangle = \langle \varphi_y, \varphi_y \rangle = e^\omega. \end{cases}$$

Let us define the quantities:

$$\begin{aligned} E &= \langle \varphi_x, \varphi_x \rangle, \quad F = \langle \varphi_x, \varphi_y \rangle, \quad G = \langle \varphi_y, \varphi_y \rangle, \\ l &= \langle \varphi_{xx}, N \rangle, \quad m = \langle \varphi_{xy}, N \rangle, \quad n = \langle \varphi_{yy}, N \rangle. \end{aligned}$$

Then the mean curvature H can be computed by the well-known classical formula¹

$$(4) \quad \begin{aligned} H &= \frac{Gl + En - 2Fm}{2(EG - F^2)} \\ &= \frac{1}{2} e^{-\omega} \langle \square \varphi, N \rangle, \end{aligned}$$

¹This formula is still valid for spacelike or timelike surfaces in Minkowski 3-space.

where \square is the *d'Alembertian operator*

$$\square = -\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

Let $u := x + y$ and $v = -x + y$. Then (u, v) defines a null coordinate system with respect to the conformal structure \mathcal{C}_I . The first fundamental form I is written in terms of (u, v) as

$$I = e^\omega du dv.$$

The partial derivatives φ_u and φ_v are computed to be

$$\varphi_u = \frac{1}{2}(\varphi_x + \varphi_y), \quad \varphi_v = \frac{1}{2}(-\varphi_x + \varphi_y).$$

In terms of null coordinates (u, v) ,

$$\langle \varphi_u, N \rangle = \langle \varphi_v, N \rangle = 0,$$

and the conformality condition (3) can be written as

$$(5) \quad \begin{cases} \langle \varphi_u, \varphi_u \rangle = \langle \varphi_v, \varphi_v \rangle = 0, \\ \langle \varphi_u, \varphi_v \rangle = \frac{1}{2}e^\omega. \end{cases}$$

The mean curvature formula (4) is written as

$$(6) \quad H = 2e^{-\omega} \langle \varphi_{uv}, N \rangle.$$

On a simply connected null coordinate region \mathbb{D} , we can find an orthonormal frame field $\mathcal{F} : \mathbb{D} \longrightarrow \mathrm{O}^{++}(3, 1)$ given by

$$\begin{aligned} \mathcal{F} &= (e^{-\omega/2}\varphi_x, e^{-\omega/2}\varphi_y, N) \\ &= (e^{-\omega/2}(\varphi_u - \varphi_v), e^{-\omega/2}(\varphi_u + \varphi_v), N). \end{aligned}$$

Here, $\mathrm{O}^{++}(3, 1)$ denotes the identity component of the Lorentz group $\mathrm{O}(3, 1)$.

The special linear group $\mathrm{SL}(2, \mathbb{R})$ acts isometrically on \mathbb{E}_1^3 via the Ad-action:

$$\mathrm{Ad} : \mathrm{SL}(2, \mathbb{R}) \times \mathbb{E}_1^3 \longrightarrow \mathbb{E}_1^3; \quad \mathrm{Ad}(g)X = gXg^{-1}.$$

The Ad-action induces a double covering $\mathrm{SL}(2, \mathbb{R}) \longrightarrow \mathrm{O}^{++}(3, 1)$.

Using this double covering, we can find a lift Φ (called a *coordinate frame*) of \mathcal{F} to $\mathrm{SL}(2, \mathbb{C})$:

$$\mathrm{Ad}(\Phi)(\mathbf{i}, \mathbf{j}', \mathbf{k}') = \mathcal{F}.$$

$\mathfrak{s} := (\varphi_u, \varphi_v, N)$ defines a moving frame on M and satisfies the following Gauß-Weingarten equations:

$$(7) \quad \mathfrak{s}_u = \mathfrak{s}\mathcal{U}, \quad \mathfrak{s}_v = \mathfrak{s}\mathcal{V},$$

where

$$\mathcal{U} = \begin{pmatrix} \omega_u & 0 & -H \\ 0 & 0 & -2e^{-\omega}Q \\ Q & \frac{1}{2}e^\omega H & 0 \end{pmatrix}, \quad \mathcal{V} = \begin{pmatrix} 0 & 0 & -2e^{-\omega}R \\ 0 & \omega_v & -H \\ \frac{1}{2}e^\omega H & R & 0 \end{pmatrix}.$$

Here, $Q := \langle \varphi_{uu}, N \rangle$ and $R := \langle \varphi_{vv}, N \rangle$. The quadratic differential 1-form

$$(8) \quad \mathcal{Q} = Qdu^2 + Rdv^2$$

is called *Hopf differential*². The second fundamental form II of M is related to \mathcal{Q} by

$$(9) \quad II = \mathcal{Q} + HI.$$

This formula implies that $p \in M$ is an *umbilic point* if and only if $\mathcal{Q}(p) = 0$, i.e., p is a common zero of Q and R .

If K denotes the Gaussian curvature, the Gauß equation which describes a relationship between K , H , Q , and R takes the following form:

$$(10) \quad H^2 - K = 4e^{-2\omega}QR.$$

The integrability condition for the Gauß-Weingarten equations is

$$(11) \quad \mathcal{V}_u - \mathcal{U}_v + [\mathcal{U}, \mathcal{V}] = 0,$$

which is equivalent to the Gauß-Codazzi equations

$$(12) \quad \omega_{uv} + \frac{1}{2}H^2e^\omega - 2e^{-\omega}QR = 0$$

$$(13) \quad H_u = 2e^{-\omega}Q_v, \quad H_v = 2e^{-\omega}R_u.$$

It follows from the Codazzi equations (13) that mean curvature H is constant if and only if $Q_v = R_u = 0$, i.e., $Q = Q(u)$ and $R = R(v)$. In this case, Q and R are said to be *Lorentz holomorphic* and *Lorentz anti-holomorphic*, respectively. For more details about Lorentz holomorphicity, see [7] and [8].

Let $\Phi : \mathbb{D} \longrightarrow \mathrm{SL}(2, \mathbb{R})$ be a coordinate frame of φ . Then the partial derivatives φ_u and φ_v are given in terms of the coordinate frame Φ as

$$\begin{aligned} \varphi_u &= \frac{1}{2}(\varphi_x + \varphi_y) \\ &= \frac{1}{2}e^{\omega/2}\mathrm{Ad}(\mathbf{i} + \mathbf{j}') \\ &= e^{\omega/2}\Phi \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \Phi^{-1} \end{aligned}$$

and

$$\begin{aligned} \varphi_v &= \frac{1}{2}(-\varphi_x + \varphi_y) \\ &= \frac{1}{2}e^{\omega/2}\mathrm{Ad}(-\mathbf{i} + \mathbf{j}') \\ &= e^{\omega/2}\Phi \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \Phi^{-1}. \end{aligned}$$

²This definition of Hopf differential was suggested to the author by J. Inoguchi [5].

It follows from Gauß-Weingarten equations (7) that the coordinate frame Φ satisfies the *Lax equations*:

$$(14) \quad \Phi_u = \Phi U, \quad \Phi_v = \Phi V,$$

where

$$U = \begin{pmatrix} \frac{\omega_u}{4} & \frac{H}{2}e^{\omega/2} \\ -Qe^{-\omega/2} & -\frac{\omega_u}{4} \end{pmatrix} \text{ and } V = \begin{pmatrix} -\frac{\omega_v}{4} & Re^{-\omega/2} \\ -\frac{H}{2}e^{\omega/2} & \frac{\omega_v}{4} \end{pmatrix}.$$

2. WEIERSTRASS REPRESENTATION FORMULA OF TIMELIKE SURFACES OF CONSTANT MEAN CURVATURE IN MINKOWSKI 3-SPACE

In this section, we derive the Weierstraß representation formula of timelike surfaces of constant mean curvature in Minkowski 3-space \mathbb{E}_1^3 .

Let M be a connected orientable 2-manifold with globally defined null coordinates system (u, v) . Let $\Phi : M \rightarrow \text{SL}(2, \mathbb{R})$ be a solution to the Lax equations (14).

Let $\hat{\Phi} := e^{\frac{\omega}{4}}\Phi$. Then $\det \hat{\Phi} = e^{\frac{\omega}{2}}$. Now, we have the following Lax equations in terms of $\hat{\Phi}$:

$$(15) \quad \frac{\partial \hat{\Phi}}{\partial u} = \hat{\Phi} \begin{pmatrix} \frac{\omega_u}{2} & \frac{H}{2}e^{\omega/2} \\ -Qe^{-\omega/2} & 0 \end{pmatrix}, \quad \frac{\partial \hat{\Phi}}{\partial v} = \hat{\Phi} \begin{pmatrix} 0 & Re^{-\omega/2} \\ -\frac{H}{2}e^{\omega/2} & \frac{\omega_v}{2} \end{pmatrix}.$$

It follows from the Lax equations (15) that

Proposition 1. *The conformal frame $\hat{\Phi}$ satisfies the Dirac equation*

$$(16) \quad e^{-\omega/2} \begin{pmatrix} 0 & \partial_u \\ -\partial_v & 0 \end{pmatrix} \hat{\Phi}^T = \frac{1}{2} H \hat{\Phi}^T.$$

Conversely,

Theorem 2 (Weierstraß Representation Formula). *Let $(s_1, -t_2)^T, (t_1, s_2)^T : M \rightarrow \mathbb{E}_1^2(u, v)$ be solutions to the Dirac equations with the potential $p \in C^\infty(M)$:*

$$(17) \quad \begin{pmatrix} 0 & \partial_u \\ -\partial_v & 0 \end{pmatrix} \begin{pmatrix} s_1 \\ -t_2 \end{pmatrix} = p \begin{pmatrix} s_1 \\ -t_2 \end{pmatrix}, \quad \begin{pmatrix} 0 & \partial_u \\ -\partial_v & 0 \end{pmatrix} \begin{pmatrix} t_1 \\ s_2 \end{pmatrix} = p \begin{pmatrix} t_1 \\ s_2 \end{pmatrix}.$$

Then $\hat{\Phi} := \begin{pmatrix} s_1 & -t_2 \\ t_1 & s_2 \end{pmatrix} : M \rightarrow \mathbb{H}'_$ is a conformal frame of the conformally immersed timelike surface $\varphi = (\varphi_1, \varphi_2, \varphi_3) : M \rightarrow \mathbb{E}_1^3$, where*

$$(18) \quad \begin{aligned} \varphi_1 &= \frac{1}{2} \int (s_1^2 + t_1^2) du - (s_2^2 + t_2^2) dv, \\ \varphi_2 &= \frac{1}{2} \int (s_1^2 - t_1^2) du + (s_2^2 - t_2^2) dv, \\ \varphi_3 &= \int (-s_1 t_1 du - s_2 t_2 dv). \end{aligned}$$

The metric ds_φ^2 and the mean curvature H of φ are given by

$$ds_\varphi^2 = (s_1 s_2 + t_1 t_2)^2 du dv, \quad H = 2pe^{-\omega/2}.$$

Proof. The partial derivatives φ_u and φ_v are given in terms of $\hat{\Phi}$ by

$$\varphi_u = e^{\frac{\omega}{2}} \hat{\Phi} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \hat{\Phi}^{-1}, \quad \varphi_v = e^{\frac{\omega}{2}} \hat{\Phi} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \hat{\Phi}^{-1}.$$

Let $\hat{\Phi} = \begin{pmatrix} s_1 & -t_2 \\ t_1 & s_2 \end{pmatrix}$ with $\det \hat{\Phi} = e^{\omega/2}$. Then

$$\begin{aligned} d\varphi &= \varphi_u du + \varphi_v dv \\ &= e^{\frac{\omega}{2}} \hat{\Phi} \left\{ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} du + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} dv \right\} \hat{\Phi}^{-1} \\ &= e^{\frac{\omega}{2}} \hat{\Phi} \begin{pmatrix} 0 & du \\ dv & 0 \end{pmatrix} \hat{\Phi}^{-1} \\ &= \begin{pmatrix} -s_1 t_1 du - s_2 t_2 dv & s_1^2 du - t_2^2 dv \\ -t_1^2 du + s_2^2 dv & s_1 t_1 du + s_2 t_2 dv \end{pmatrix}. \end{aligned}$$

Now, φ can be retrieved by integrating $d\varphi$:

$$\begin{aligned} \varphi &= \int d\varphi \\ &= \int \begin{pmatrix} -s_1 t_1 du - s_2 t_2 dv & s_1^2 du - t_2^2 dv \\ -t_1^2 du + s_2^2 dv & s_1 t_1 du + s_2 t_2 dv \end{pmatrix}. \end{aligned}$$

On the other hand, by the identification (1),

$$\varphi = (\varphi_1, \varphi_2, \varphi_3) \cong \begin{pmatrix} \varphi_3 & \varphi_1 + \varphi_2 \\ -\varphi_1 + \varphi_2 & -\varphi_3 \end{pmatrix}.$$

Hence, we obtain

$$\begin{aligned} \varphi_1 &= \frac{1}{2} \int (s_1^2 + t_1^2) du - (s_2^2 + t_2^2) dv, \\ \varphi_2 &= \frac{1}{2} \int (s_1^2 - t_1^2) du + (s_2^2 - t_2^2) dv, \\ \varphi_3 &= \int (-s_1 t_1 du - s_2 t_2 dv). \end{aligned}$$

□

3. THE GAUSS MAP AND WEIERSTRASS REPRESENTATION FORMULA

In the previous section, we obtained Weierstraß representation formula for timelike surfaces in Minkowski 3-space. In this section, we study the relationship between the data s_1, s_2, s_1, t_2 and the Gauß map of timelike surface which is given by the Weierstraß representation formula.

The pseudosphere of radius 1 in Minkowski 3-space is the hyperquadric

$$S_1^2 = \{(x_1, x_2, x_3) \in \mathbb{E}_1^3 : -x_1^2 + x_2^2 + x_3^2 = 1\}$$

of constant Gaußian curvature 1.

Let $\varphi : M \longrightarrow \mathbb{E}_1^3$ be an orientable timelike surface and N a unit normal vector field to φ . This unit normal vector field $p \in M \longmapsto N(p) \in S_1^2$ is defined to be the *Gauß map* of φ .

The Ad-action of $\mathrm{SL}(2, \mathbb{R})$ on S_1^2 is transitive and isometric. The isotropy subgroup at \mathbf{k}' is $\mathrm{SO}(1, 1)$. Thus, S_1^2 is identified with the homogeneous space

$$\mathrm{SL}(2, \mathbb{R})/\mathrm{SO}(1, 1) = \left\{ h\mathbf{k}'h^{-1} : h \in \mathrm{SL}(2, \mathbb{R}) \right\}.$$

Let $\wp_{\mathcal{N}} : S_1^2 \setminus \{x_3 = 1\} \longrightarrow \mathbb{E}_1^2 \setminus H_0^1$ be the stereographic projection from the north pole $\mathcal{N} = (0, 0, 1)$. Here, $H_0^1 = \{(x_1, x_2) \in \mathbb{E}_1^2 : -x_1^2 + x_2^2 = -1\}$. Then

$$\begin{aligned} \wp_{\mathcal{N}}(x_1, x_2, x_3) &= \left(\frac{x_1}{1-x_3}, \frac{x_2}{1-x_3} \right) \\ &\cong \left(\frac{x_1+x_2}{1-x_3}, \frac{-x_1+x_2}{1-x_3} \right) \text{ in null coordinate system } (u, v). \end{aligned}$$

The inverse stereographic projection $\wp_{\mathcal{N}}^{-1} : \mathbb{E}_1^2 \setminus H_0^1 \longrightarrow S_1^2 \setminus \{x_3 = 1\}$ is given by

$$\wp_{\mathcal{N}}^{-1}(x, y) = \left(\frac{2x}{1-x^2+y^2}, \frac{2y}{1-x^2+y^2}, \frac{-1-x^2+y^2}{1-x^2+y^2} \right).$$

In terms of null coordinates (u, v) ,

$$(19) \quad \wp_{\mathcal{N}}^{-1}(x, y) = \wp_{\mathcal{N}}^{-1} \left(\frac{u-v}{2}, \frac{u+v}{2} \right) = \left(\frac{u-v}{1+uv}, \frac{u+v}{1+uv}, \frac{-1+uv}{1+uv} \right).$$

The Gauß map can be written as $N = h \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} h^{-1} \in S_1^2$ for some $h = \begin{pmatrix} p_1 & -q_2 \\ q_1 & p_2 \end{pmatrix} \in \mathrm{SL}(2, \mathbb{R})$. That is, $N = \begin{pmatrix} p_1p_2 - q_1q_2 & 2p_1q_2 \\ 2p_2q_1 & -p_1p_2 + q_1q_2 \end{pmatrix}$ and its projected image via $\wp_{\mathcal{N}}$ is $\wp_{\mathcal{N}}(N) \cong \left(\frac{p_1}{q_1}, \frac{p_2}{q_2} \right)$ in null coordinate system (u, v) .

Now, $\hat{\Phi} = \begin{pmatrix} s_1 & -t_2 \\ t_1 & s_2 \end{pmatrix} = e^{\frac{\omega}{2}} \begin{pmatrix} p_1 & -q_2 \\ q_1 & p_2 \end{pmatrix}$ and $\begin{pmatrix} p_1 & -q_2 \\ q_1 & p_2 \end{pmatrix} \in \mathrm{SL}(2, \mathbb{R})$. Then,

$$q := \frac{p_1}{q_1} = \frac{s_1}{t_1} \text{ and } r := \frac{p_2}{q_2} = \frac{s_2}{t_2}.$$

Therefore, the Weierstraß representation formula (18) can be written in terms of the projected Gauß map (q, r) as

$$\begin{aligned}
 \varphi_1 &= \frac{1}{2} \int (1 + q^2) f(u) du - (1 + r^2) g(v) dv, \\
 \varphi_2 &= -\frac{1}{2} \int (1 - q^2) f(u) du + (1 - r^2) g(v) dv, \\
 \varphi_3 &= - \int q f(u) du + r g(v) dv,
 \end{aligned}
 \tag{20}$$

where $f(u) = t_1^2$ and $g(v) = t_2^2$. The metric is given by

$$ds_\varphi^2 = (1 + qr)^2 f(u) g(v) du dv. \tag{21}$$

Using the Lax equations (15), we compute

Proposition 3. *The projected Gauß map $\wp_N \circ N$ satisfy the following equations:*

$$q_u = \frac{Q}{f(u)}, \tag{22}$$

$$q_v = \frac{He^\omega}{2f(u)} = \frac{H}{2} (1 + qr)^2 g(v), \tag{23}$$

$$r_u = \frac{He^\omega}{2g(v)} = \frac{H}{2} (1 + qr)^2 f(u), \tag{24}$$

$$r_v = \frac{R}{g(v)}. \tag{25}$$

Remark 1. In [8], the authors studied the *normalized Weierstraß formula* for timelike minimal surfaces via *loop group method*. In this case, $f(u) = g(v) = 1$ and so, by equations (22) and (25), q and r are interpreted as the primitive functions of the coefficients Q and R , resp. of the Hopf differential (8). Originally, the Weierstraß formula (20) was obtained by M. A. Magid in [6], however the geometric meaning of the data (q, r) is not clarified. In [8], the data (q, r) are retrieved from the *normalized potential* in their construction.

The corollaries 4 and 5 immediately follow from the equations (22)-(25).

Corollary 4. *A Lorentz surface $\varphi : M \longrightarrow \mathbb{E}_1^2$ is minimal (i.e., $H = 0$) if and only if q is Lorentz holomorphic and r is Lorentz antiholomorphic.*

Corollary 5. *A Lorentz surface $\varphi : M \longrightarrow \mathbb{E}_1^2$ is totally umbilic if and only if q is Lorentz antiholomorphic and r is Lorentz holomorphic.*

Corollary 6. *A totally umbilic minimal Lorentz surface in \mathbb{E}_1^3 is part of timelike plane in \mathbb{E}_1^3 .*

Proof. By the equation (9), a totally umbilic minimal Lorentz surface has $II = 0$, i.e., it is totally geodesic. \square

Theorem 7 (Weierstraß Representation Formula for Timelike Minimal Surfaces in \mathbb{E}_1^3). *Let $q, r : M \longrightarrow \mathbb{E}_1^2$ be Lorentz holomorphic and Lorentz antiholomorphic maps, resp. Then*

$$(26) \quad \begin{aligned} \varphi_u &= \left(\frac{1}{2}(1+q^2), -\frac{1}{2}(1-q^2), -q \right) f(u), \\ \varphi_v &= \left(-\frac{1}{2}(1+r^2), -\frac{1}{2}(1-r^2), -r \right) g(v) \end{aligned}$$

define a timelike minimal surface $\varphi : M \longrightarrow \mathbb{E}_1^3$. Here, $f(u)$ and $g(v)$ are Lorentz holomorphic and Lorentz antiholomorphic maps. The metric of φ is given by $ds_\varphi^2 = (1+qr)^2 f(u)g(v)dudv$. Conversely, any timelike minimal surface can be represented by (26) up to translations.

Proof. The first statement follows immediately from (20), (21), (23), and (24).

Let $\varphi : M \longrightarrow \mathbb{E}_1^3$ be a timelike minimal surface parametrized by null coordinates (u, v) . Define two vector valued functions $\xi = (\xi_0, \xi_1, \xi_2)$ and $\eta = (\eta_0, \eta_1, \eta_2)$ by

$$\xi(u) := \varphi_u, \quad \eta(v) := \varphi_v,$$

that is,

$$X(u) = \int_0^u \xi(u)du, \quad Y(v) = \int_0^v \eta(v)dv$$

for the timelike minimal surface $\varphi(u, v) = X(u) + Y(v)$. Since φ is a solution to the wave equation $\varphi_{uv} = 0$, ξ and η are a Lorentz holomorphic and a Lorentz anti-holomorphic null curves, resp., in \mathbb{E}_1^3 . Hence, they satisfy

$$-\xi_0^2 + \xi_1^2 + \xi_2^2 = -\eta_0^2 + \eta_1^2 + \eta_2^2 = 0.$$

Define the functions $q(u), f(u), r(v)$ and $g(v)$ by

$$\begin{aligned} -\xi_0 + \xi_1 &= -f, \quad \xi_2 = -qf, \\ \eta_0 + \eta_1 &= -g, \quad \eta_2 = -rg. \end{aligned}$$

Then we obtain

$$\begin{aligned} \xi(u) &= \left(\frac{1}{2}(1+q(u)^2), -\frac{1}{2}(1-q(u)^2), -q(u) \right) f(u), \\ \eta(v) &= \left(-\frac{1}{2}(1+r(v)^2), -\frac{1}{2}(1-r(v)^2), -r(v) \right) g(v). \end{aligned}$$

Thus, the given timelike minimal surface is represented by

$$(27) \quad \begin{aligned} \varphi(u, v) &= \int_0^u \left(\frac{1}{2}(1+q(u)^2), -\frac{1}{2}(1-q(u)^2), -q(u) \right) f(u)du \\ &\quad + \int_0^v \left(-\frac{1}{2}(1+r(v)^2), -\frac{1}{2}(1-r(v)^2), -r(v) \right) g(v)dv \end{aligned}$$

up to translations. □

Remark 2. The maps q and r coincide with the first and the second component maps of the projected Gauß map $\wp_N \circ N$ of the timelike minimal surface φ .

Example 1 (Lorentz Enneper surfaces). Let $q(u) = \varepsilon u = \pm u$, $f(u) = 1$, $r(v) = v$, $g(v) = 1$. Then we obtain the following timelike minimal immersion

$$\varphi^{(\varepsilon)}(u, v) = X(u) + Y(v),$$

where

$$\begin{aligned} X(u) &= \frac{1}{2} \left(u + \frac{u^3}{3}, -u + \frac{u^3}{3}, \mp u^2 \right), \\ Y(v) &= \frac{1}{2} \left(-v - \frac{v^3}{3}, -v + \frac{v^3}{3}, -v^2 \right). \end{aligned}$$

The timelike minimal surface $\varphi^{(\varepsilon)}$ is called *Lorentz Enneper surface*.

The metric of $\varphi^{(\varepsilon)}$ is given by

$$I = (1 + \varepsilon uv)^2 du dv.$$

The Hopf differential and the Gaussian curvature of $\varphi^{(\varepsilon)}$ are

$$\varepsilon du^2 + dv^2, \quad K = -4\varepsilon(1 + \varepsilon uv)^{-4}.$$

The surface $\varphi^{(-1)}$ has two imaginary principal curvatures, while $\varphi^{(1)}$ has real distinct principal curvatures.

Definition 1. Let $\varphi(u, v) = X(u) + Y(v)$ be a timelike minimal surface. Then clearly, $\hat{\varphi}(u, v) := X(u) - Y(v)$ is also a timelike minimal surface. The timelike minimal surface $\hat{\varphi}$ is called the conjugate timelike minimal surface of φ .

Example 2 (Lorentz catenoid and Lorentz helicoid with a spacelike axis). *Lorentz catenoids* are timelike minimal surfaces of revolution with a spacelike axis or timelike axis. Lorentz catenoid with a spacelike axis $\varphi(u, v) = X(u) + Y(v)$ (Figure 2(a)) can be obtained by the Weierstraß formula (26) with data $q(u) = -e^u$, $f(u) = -e^{-u}$, $r(v) = e^{-v}$, $g(v) = -e^v$, where

$$X(u) = (-\sinh u, -\cosh u, -u), \quad Y(v) = (\sinh v, \cosh v, v).$$

The conjugate surface $\hat{\varphi} = X(u) - Y(v)$ is called *Lorentz helicoid* with a spacelike axis (Figure 2(b)).

The ordered pair $(q, r) = (-e^u, e^{-v})$ is the Gauß map projected in $\mathbb{E}_1^2(u, v)$. Note that Lorentz catenoid and Lorentz helicoid with a spacelike axis share the same Gauß map analogously to the Euclidean case.

Example 3 (Lorentz catenoid and Lorentz helicoid with timelike axis). Lorentz catenoid with timelike axis $\varphi(u, v) = X(u) + Y(v)$ (Figure 3(a)) can be obtained by the Weierstraß formula (26) with data $q(u) = \frac{\sin u}{-1 + \cos u}$, $f(u) = -1 + \cos u$, $r(v) = \frac{\sin v}{1 + \cos v}$, $g(v) = -(1 + \cos v)$, where

$$X(u) = (-u, -\sin u, \cos u), \quad Y(v) = (v, \sin v, -\cos v).$$

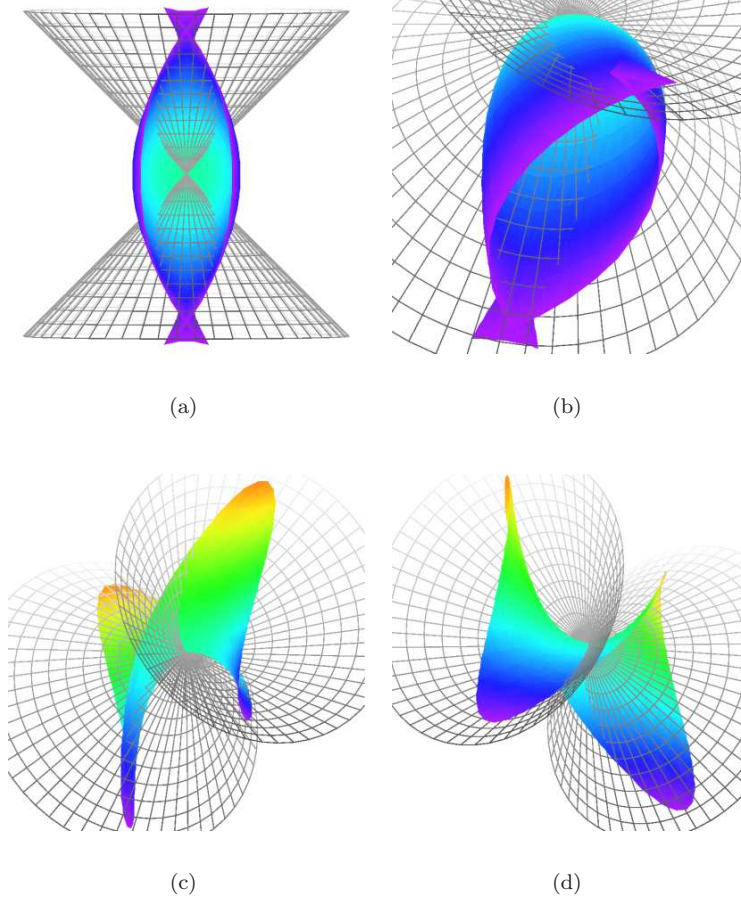


FIG. 1. Lorentz Enneper surfaces $\varphi^{(1)}$ (a), (b) and $\varphi^{(-1)}$ (c), (d)

The conjugate surface $\hat{\varphi} = X(u) - Y(v)$ is called *Lorentz helicoid* with timelike axis (Figure 3(b)).

The ordered pair $(q, r) = \left(\frac{\sin u}{-1 + \cos u}, \frac{\sin v}{1 + \cos v} \right)$ is the Gauß map projected in $\mathbb{E}_1^2(u, v)$. Note that timelike catenoid and timelike helicoid with timelike axis share the same Gauß map analogously to the Euclidean case.

It is known that a totally umbilic Lorentz surface in \mathbb{E}_1^3 is congruent to an open part of the pseudosphere S_1^2 or timelike plane in \mathbb{E}_1^3 . This property can be proved directly from the Weierstraß formula (20).

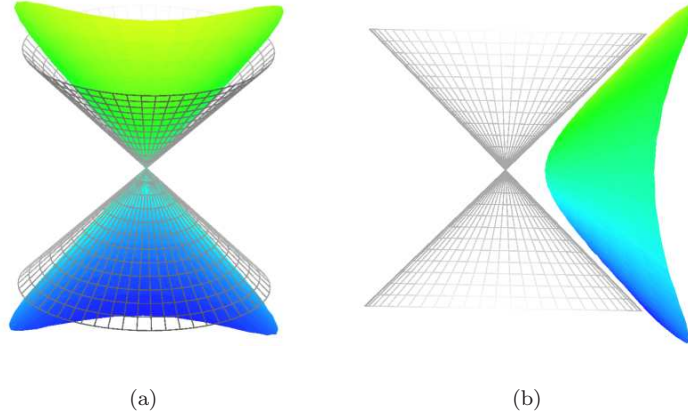


FIG. 2. Lorentz catenoid and Lorentz helicoid with a space-like axis

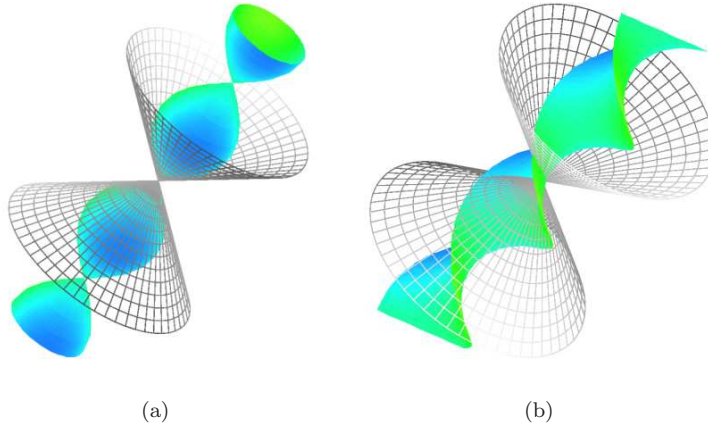


FIG. 3. Lorentz catenoid and Lorentz helicoid with timelike axis

Corollary 8. *A totally umbilic timelike surface in \mathbb{E}_1^3 is part of timelike plane or a pseudosphere in \mathbb{E}_1^3 . In particular, a totally umbilic timelike surface with positive Gaussian curvature K is part of a pseudosphere in \mathbb{E}_1^3 of radius $\frac{1}{\sqrt{K}}$.*

Proof. Let M be a simply connected domain that contains the origin $(0, 0)$. Let $\varphi = (\varphi_1, \varphi_2, \varphi_3) : M \longrightarrow \mathbb{E}_1^3$ be a totally umbilic timelike surface in \mathbb{E}_1^3 .

If φ is minimal, by Corollary 6 it is part of timelike plane. So, we assume that φ is not minimal, i.e., $H \neq 0$.

By the Weierstrß formula (20),

$$\begin{aligned}\varphi_1 &= \int \frac{1}{2}(1+q^2)f(u)du - \frac{1}{2}(1+r^2)g(v)dv \\ &= \int \frac{1+q^2}{H(1+qr)^2}dr - \frac{1+r^2}{H(1+qr)^2}dq.\end{aligned}$$

Since $\frac{\partial \varphi_1}{\partial r} = \frac{1+q^2}{H(1+qr)^2}$,

$$\begin{aligned}\varphi_1 &= \int_0^r \frac{1+q^2}{H(1+qr)^2}dr + \psi(q) \\ &= -\frac{1+q^2}{H(1+qr)q} + \psi(q).\end{aligned}$$

So,

$$\frac{\partial \varphi_1}{\partial q} = \frac{-q^2 + 2qr + 1}{(1+qr)^2} + \psi'(q).$$

This must be the same as $-\frac{1+r^2}{H(1+qr)^2}$; hence $\psi'(q) = -\frac{1}{Hq^2}$, that is, $\psi(q) = \frac{q}{Hq} + c_1$, where c_1 is a constant. Thus, φ_1 is given by

$$\varphi_1 = -\frac{q-r}{H(1+qr)} + c_1.$$

Similarly, we compute

$$\begin{aligned}\varphi_2 &= -\frac{q+r}{H(1+qr)} + c_2, \\ \varphi_3 &= \frac{1-qr}{H(1+qr)} + c_3,\end{aligned}$$

where c_2 and c_3 are constants. The parametric equation

$$\varphi = \left(-\frac{q-r}{H(1+qr)} + c_1, -\frac{q+r}{H(1+qr)} + c_2, \frac{1-qr}{H(1+qr)} + c_3 \right)$$

shows that φ is part of the pseudosphere in \mathbb{E}_1^3 centered at (c_1, c_2, c_3) with radius $\frac{1}{H}$. (See the equation (19).) Since φ is totally umbilic, $Q = R = 0$. So, by the equation (10), $H^2 = K$. If the Gaussian curvature K is positive, then $H = \sqrt{K}$. This completes the proof. \square

4. APPENDIX: TIMELIKE MINIMAL SURFACES AND BOSONIC NAMBU-GOTO STRINGS IN MINKOWSKI SPACETIME

By the equation (6), a Lorentz surface $\varphi : M \longrightarrow \mathbb{E}_1^3$ is minimal if and only if it is a solution to the homogeneous *wave equation* $\varphi_{uv} = 0$ (or equivalently $\square\varphi = 0$). Hence, as seen in (20), timelike minimal surfaces in Minkowski $(2+1)$ -spacetime can be retrieved by integrating a pair of Lorentz holomorphic and Lorentz antiholomorphic null curves in spacetime, i.e., curves in the light cone $\mathcal{N} = \{(x_1, x_2, x_3) \in \mathbb{E}_1^3 : -x_1^2 + x_2^2 + x_3^2 = 0\}$. Note that Lorentz holomorphic and Lorentz antiholomorphic null curves in spacetime are the trajectories of massless particles (*bosonic* particles) in spacetime. Therefore, there appears to be some relationship between timelike minimal surfaces and bosonic particles in spacetime. In string theory, elementary particles are considered to be tiny vibrating strings in spacetime. A string evolves in time while sweeping a surface, the so-called *worldsheet*, in spacetime. Hence, string worldsheets are timelike surfaces. Moreover, it can be shown that string worldsheets are indeed timelike minimal surfaces in spacetime. The following discussion is not restricted in $(2+1)$ -dimensions. So, one can assume the standard $(3+1)$ -spacetime or any higher $(D+1)$ -dimensional spacetime. (Here, we consider only classical strings. In case of *quantized* strings, our universe is, without supersymmetry, a 26-dimensional spacetime and it is a 10-dimensional spacetime with supersymmetry.)

In order to be consistent with physicists' common notations, we consider string worldsheets parametrized by τ and σ , where τ is time parameter in $(1+1)$ -spacetime. The motion of bosonic strings in spacetime is described by the *Nambu-Goto string action*

$$(28) \quad \mathcal{S} = -T \int (-\det h_{ab})^{1/2} d\tau d\sigma,$$

where T is tension and h_{ab} is the metric tensor of the worldsheet $\varphi : M \longrightarrow \mathbb{E}_1^D$. Using the *Einstein's convention*, h_{ab} is given by

$$h_{ab} = \partial_a \varphi^\mu \partial_b \varphi^\nu \eta_{\mu\nu},$$

where $\eta_{\mu\nu}$ is the metric tensor of the flat Minkowski $(D+1)$ -spacetime with signature $(-, +, \dots, +)$. Clearly, $dA := (-\det h_{ab})^{1/2} d\tau d\sigma$ is the area element of the string worldsheet φ .

Let us denote $\dot{\varphi} := \frac{\partial \varphi}{\partial \tau}$ and $\varphi' := \frac{\partial \varphi}{\partial \sigma}$. The *Lagrangian* \mathcal{L} of the string motion is

$$(29) \quad \mathcal{L}(\dot{\varphi}, \varphi'; \sigma, \tau) = -T(-\det h_{ab})^{1/2}.$$

By variational principle, $\delta\mathcal{S} = 0$, subject to the condition that the initial and final configurations of the string are kept fixed, implies the *Euler-Lagrange*

equation for the string action (28) is

$$(30) \quad \frac{\partial}{\partial \tau} \frac{\partial \mathcal{L}}{\partial \dot{\varphi}^\mu} + \frac{\partial}{\partial \sigma} \frac{\partial \mathcal{L}}{\partial \varphi'^\mu} = 0$$

with

$$\frac{\partial \mathcal{L}}{\partial \dot{\varphi}^\mu} = \frac{\partial \mathcal{L}}{\partial \varphi'^\mu} = 0 \text{ on } \partial M.$$

Note that the action (28) is invariant under conformal scaling of the worldsheet metric. Physicists call it *Weyl invariance* and it is an important symmetry of the action (28) along with *worldsheet reparametrizations* and *Lorentz/Poincaré symmetries*. With *conformal gauge fixing*

$$-\langle \varphi_\tau, \varphi_\tau \rangle = \langle \varphi_\sigma, \varphi_\sigma \rangle = e^\omega, \quad \langle \varphi_\tau, \varphi_\sigma \rangle = 0,$$

one can easily show that the Euler-Lagrange equation (30) is equivalent to the homogeneous wave equation

$$\square \varphi = -\frac{\partial^2 \varphi}{\partial \tau^2} + \frac{\partial^2 \varphi}{\partial \sigma^2} = 0.$$

Hence, we see that string worldsheets are timelike minimal surfaces.

The Gaußian curvature K also plays an important role in string theory, especially when results from different orders of string perturbation theory are compared and when string interactions are considered. In order to write a more familiar expression in string theory, we introduce a new parameter α' which is defined by

$$\alpha' = \frac{1}{2\pi T}.$$

This parameter α' is known as the slope of *Regge trajectories* in physics. We refer to [9], [10] for physical background and details regarding Regge trajectories. As is well-known in variational theory, one can add more terms to action functionals as constraints. One physically interesting extra term is the *Einstein-Hilbert action*

$$(31) \quad \chi := \frac{1}{4\pi\alpha'} \int_M R(-\det h_{ab})^{1/2} d\tau d\sigma + \frac{1}{2\pi\alpha'} \int_{\partial M} k ds,$$

where R is the Ricci scalar (or scalar curvature) on the worldsheet M and k is the geodesic curvature on ∂M . Note that the Einstein-Hilbert action (31) is invariant under the conformal transformation (Weyl transformation) $h_{ab} \longrightarrow e^\omega h_{ab}$. Now, the full string action is given by

$$\begin{aligned} \mathcal{S}' = & -\frac{1}{2\pi\alpha'} \int (-\det h_{ab})^{1/2} d\tau d\sigma + \lambda \left\{ \frac{1}{4\pi\alpha'} \int_M R(-\det h_{ab})^{1/2} d\tau d\sigma \right. \\ & \left. + \frac{1}{2\pi\alpha'} \int_{\partial M} k ds \right\}, \end{aligned}$$

where λ is a coupling parameter. This full string action resembles 2-dim gravity coupled with bosonic matter fields and the equations of motion is given by the following Einstein's field equation:

$$R_{ab} - \frac{1}{2}h_{ab}R = T_{ab}.$$

The LHS of the Einstein's equation vanishes identically in 2-dimensions, so there is no dynamics associated with (31). For surfaces, the Ricci scalar R and the Gaußian curvature K are related by $R = 2K$, thus (31) can be written as

$$(32) \quad \chi := \frac{1}{2\pi\alpha'} \int_M K dA + \frac{1}{2\pi\alpha'} \int_{\partial M} k ds.$$

This is nothing but a constant multiple $\left(\frac{1}{2\pi\alpha'}\right)$ of the *Gauß-Bonnet term*. In string theory, physicists apply the so-called *Wick rotation* $\tau \longrightarrow i\tau$ to change Lorentzian signature to Euclidean signature. As a result, string worldsheets turn into Riemann surfaces with complex local coordinates $\tau \pm i\sigma$. This procedure is required because string amplitudes are computed by the Feynman path integral which is defined in Euclidean setting. Once calculation is done, one retrieves the Lorentzian signature by the opposite Wick rotation for a physical interpretation. In Euclidean setting, by the well-known *Gauß-Bonnet Theorem*, the RHS of (32) is the same as $\frac{1}{\alpha'}\chi(M)$ where $\chi(M)$ is the Euler characteristic of the string worldsheet M as a *compact Riemann surface*. String worldsheets that are swept by closed strings are compact orientable surfaces without boundary and so $\chi(M) = 2 - 2g$, where g is the genus of the (Riemannian) worldsheet M . Therefore, a closed string is distinguished from another by the genus of its (Riemannian) worldsheet, which solely depends on the topology of the worldsheet.

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